UNIT - 1

"Differential Equation"

⇒ **Exact differential equation**: A differential eqⁿ of type $M[x,y]\,dx + N[x,y]\,dy = 0$ is said to be exact if there exist a function $f[x,y]$ such that differential eqⁿ can be written as $d[f(x,y)] = 0$ in that case solution to the differential eqⁿ is $f(x,y) = C$

$$\left[\begin{aligned} M(x,y)\,dx &= -N[x,y]\,dy \\ \frac{dy}{dx} &= \frac{-M[x,y]}{N[x,y] \to \neq 0} \end{aligned}\right] \quad \frac{dy}{dx} = \phi(x,y)$$

eg. $x\,dy + y\,dx = 0$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$y\,dx + x\,dy = 0$

$\downarrow \qquad \downarrow$

$M = y \qquad N = x$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$$\frac{\partial M}{\partial y} = 1 \quad \& \quad \frac{\partial N}{\partial x} = 1$$

If $M[x,y]dx + N[x,y]dy = 0$ is exact then, general solution of the differential eq^n. is $\int M dx + \int(\text{terms of } N \text{ not containing } x)dy = C$

$y = $ Constant.

Ques: check whether the following differential equation are exact and obtain the general solution.

(i) $(1+e^x)dx + y\,dx = 0$

(ii) $[3x^2y + y/x]dx + [x^3 + \log x]dy = 0$

(iii) $(2x + e^y)dx + xe^y dy = 0$

Sol^n (i) $(1+e^x)dx + y\,dy = 0$

Compare it $M[x,y]dx + N[x,y]dy = 0$

$M = 1+e^x$ & $N = y$

Now. $\dfrac{\partial M}{\partial y} = 0 \qquad \dfrac{\partial N}{\partial x} = 0$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Given differential ∴ it is exact and solvable.

Date ___ / ___ / ___
Page ___

Now, solution of given differential eq^n

$$\int_{y = \text{Constant}} M\,dx + \int [\text{terms of } N \text{ not containing } x]\, dy = C$$

$$\Rightarrow \int (1+e^x)\,dx + \int y\,dy = C.$$

$$x + \frac{e^x}{\cancel{x}} + \frac{y^2}{2} = C.$$

(11) $\left[3x^2y + \frac{y}{x}\right]dx + \left[x^3 + \log x\right]dy = 0$

$$M = 3x^2y + \frac{y}{x} \qquad N = x^3 + \log x$$

$$\frac{\partial M}{\partial y} = \frac{1}{x} + 3x^2 \qquad \frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x}.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \left[\text{It is exact}\right].$$

$$\int \left(\cancel{\frac{1}{x} + 3x^2}\right)\left(3x^2y + \frac{y}{x}\right)dx + \int 0\,dx = 0$$

$$\frac{3x^3}{3}y + y\log x = C.$$

$$x^3y + y\log|x| = C.$$

Scanned by CamScanner

(iii) $(2x + e^y)\,dx + x e^y\,dy = 0$

$M = 2x + e^y \qquad N = e^y x$

$\frac{\partial M}{\partial y} = e^y \qquad \frac{\partial N}{\partial x} = e^y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ∴ It is exact.

$\int (2x + e^y)\,dx + \int e^y\,dy = c$

$\frac{2x^2}{2} + x e^y + e^y = c$

$x^2 + x e^y + e^y = c$

(iv) $x\,dy - y\,dx = e^y (x^2 + y^2)\,dy$

$\cancel{x\,dy}\ dy(e^y x^2 + e^y y^2 + x) + y\,dx = 0$

$M = 0 - y \qquad N = e^y x^2 + y^2 e^y + x$

$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = \cancel{2x + 1}$

$1 - e^y(2x) + \cancel{e^y}$ -0

$= 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ∴ So given differential eq. is not exact.

Integrating factor :

Let $M(x,y)dx + N(x,y)dy = 0$ is not exact then a term $\mu(x,y)$ which when multiplied with the given differential equation, make it exact, is called an integrating factor.

$$\frac{dy}{dx} + Py = Q$$

If. $I.F = e^{\int P dx}$.

Rules to find out Integrating factor:

(1) If both $M(x,y)$ & $N(x,y)$ in $Mdx + Ndy = 0$ ~~both~~ are are homogeneous function of degree n, then integrating factor. =

$$I.F = \frac{1}{Mx + Ny}, \quad Mx + Ny \neq 0.$$

Ques: Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Compare it with $Mdx + Ndy = 0$

$M = x^2y - 2xy^2$, $N = x^3 - 3x^2y$

$\frac{\partial M}{\partial y} = x^2 - 4xy$ $\quad \frac{\partial N}{\partial x} = 3x^2 - 6xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ It is not exact.

Here $M(x,y) = x^2y - 2xy^2$ &

$N(x,y) = -x^3 + 3x^2y$

Here Now, $M(\lambda x, \lambda y) = (\lambda x)^2(\lambda y) - 2(\lambda x)(\lambda y)^2$

$= \lambda^2 x^2 \lambda y - 2\lambda x \lambda^2 y^2$
$= \lambda^3 (x^2y - 2xy^2)$
$= \lambda^3 M(x,y)$

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$\therefore f(x,y)$ is homogenous with degree n.

$\therefore M(\lambda x, \lambda y) = \lambda^3 M(x,y)$

$\therefore M(x,y)$ is homogenious function with degree $= 3$.

Again, $N[\lambda x, \lambda y] = -\lambda^3 x^3 + 3\lambda^3 x^2 y = \lambda^3[-x^3 + 3x^2y]$
$= \lambda^3 N[x,y]$

$N(\lambda x, \lambda y) = \lambda^3 N(x,y)$

$\therefore N(x,y)$ is a homogenous function with degree $= 3$

$\therefore$ Integrating factor $= I.f = \dfrac{1}{Mx + Ny}$

$$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$\therefore \text{I.F} = \frac{1}{\cancel{x^2+y^2}\; x^2y^2}$$

multiply given diff. eq. (1) with $\frac{1}{x^2y^2}$

$$(x^2y - 2xy^2)\,dx - [x^3 - 3x^2y]\,dy = 0.$$

$\therefore$ After multiplying $\frac{1}{x^2y^2}$

$$\left[\frac{1}{y} - \frac{2}{x}\right]dx - \left[\frac{x}{y^2} - \frac{3}{y}\right]dy = 0$$

$$M' = \frac{1}{y} - \frac{2}{x} \qquad N' = \frac{x}{y^2} - \frac{3}{y}$$

$$\frac{dM'}{dy} = -\frac{1}{y^2} \qquad \frac{dN'}{dx} = \frac{-1}{y^2}$$

$$\frac{dM'}{dy} = \frac{dN'}{dx} \rightarrow \text{given differential eq. is exact.}$$

$\therefore$ solution ~~of~~ to eq. (11) is.

$$\int\left(\frac{1}{y}+\frac{2}{x}\right)dx+\int\frac{3}{y}\,dy$$

$$\frac{x}{y}+2\log|x|+3\log|y|=c.$$

* **Rule 2** :

If $Mdx+Ndy=0$ is of type
$f_1(x,y)\,y\,dx+f_2(xy)\,x\,dy=0$

then $I.F=\dfrac{1}{Mx-Ny}$, $Mx-Ny\neq 0$

Ques: $(x^3y^2+x)dy+(x^2y^3-y)dx=0$

$M=(x^2y^3-y)$ $\quad N=(x^3y^2+x)$

$\dfrac{\partial M}{\partial y}=3x^2y^2-1$ $\quad \dfrac{\partial N}{\partial y}=3x^2y^2+1$

$\therefore \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial y}$ $\therefore$ It is not exact.

Now, $(x^2y^2+1)x\,dy+(x^2y^2-1)y\,dy$

Date / /
Page

$$\therefore I.F = \frac{1}{Mx - Ny} = \frac{1}{(x^3y^3 - xy) - (x^3y^3 + xy)}$$

$$= \frac{1}{x^3y^3 - xy - x^3y^3 - xy} = \frac{-1}{2xy}.$$

multiply (1) with $\frac{-1}{2xy}$

$$(x^2y^2 + x)\,dy + (x^2y^3 - y)\,dx = 0$$

$$\left[-\frac{x^2y}{2} + \frac{1}{2y}\right]dy + \left[-\frac{xy^2}{2} + \frac{1}{2x}\right]dx. \quad \text{(II)}$$

$$\frac{\partial M'}{\partial y} = -\frac{2xy}{2} = \cancel{\frac{1}{2x^2}} \qquad \frac{\partial N'}{\partial x} = -\frac{2xy}{2} = \cancel{\frac{1}{2y^2}}.$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial N} \quad \{\text{It is exact}\}$$

$$\int \cancel{\left[-\frac{x^2y}{2} - \frac{1}{2y}\right]} \int \left[-\frac{xy^2}{2} + \frac{1}{2x}\right]dx + \int \frac{-1}{y}dy$$

$$-\frac{x^2y^2}{2} + \frac{1}{2}\log|x| - \frac{1}{2}\log|y| = c$$

Scanned by CamScanner

Integrating factor is not unique

* **Rule 3 :**

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ [function of x only]

then $I.F = e^{\int f(x)dx}$

* **Rule 4 :** If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$ [function of y only]

then, $I.F = e^{-\int g(y)dy}$

Ques: $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

Sol^n :

$M = y^4 + 2y \qquad N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2 \qquad \frac{\partial N}{\partial x} = y^3 - 4$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ (1) is not exact

Now, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = (4y^3 + 2) - (y^3 - 4) = 3y^3 + 6$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3y^3 + 6}{y^4 + 2y} = \frac{3}{y}$$

$e^{\log a} = a.$

Date / /
Page

$\therefore \ I.F = e^{\int \frac{3}{y} dy} = e^{-3 \log y}$

$= e^{\log y^{-3}} = \frac{1}{y^3}.$

Multiplying ① $\frac{1}{y^3}$

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$

$M' = y + \frac{2}{y^2} \qquad N' = x + 2y - \frac{4x}{y^3}$

$\frac{\partial M'}{\partial y} = 1 + \frac{-4}{y^3} \qquad \frac{\partial N'}{\partial x} = 1 - \frac{4}{y^3}.$

$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$ $\therefore$ It is exact

$\int \left(y + \frac{2}{y^2}\right)dx + \int 2y\, dy$

$xy + \frac{2}{y^2}x + \frac{2y^2}{2} = C.$

Scanned by CamScanner

Date ___ / ___ / ___
Page ___

Ques: solve $(4xy + 3y^2 - x)dx + x(x+2y)dy = 0$

Sol^n: Compare it with $Mdx + Ndy = 0$

$M = 4xy + 3y^2 - x$ $N = x(x+2xy)$

$\frac{\partial M}{\partial y} = 4x + 6y$ & $\frac{\partial N}{\partial x} = 2x + 2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ① is not exact.

Now $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x + 6y - 2x - 2y$

$= 2x + 4y = 2[x+2y]$

Now, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2[x+2y]}{x[x+2y]} = \frac{2}{x}$

Now, I.F $= e^{\int \frac{2}{x} dx}$ $e^{2\log x} = e^{\log x^2} = x^2$

Now, multiply ① with x^2.

$[4x^3y + 3x^2y^2 - x^3]dx + [x^4 + 2x^3y]dy = 0$

$M' = 4x^3y + 3x^2y^2 - x^3$ $N' = x^4 + 2x^3y$

$\frac{M'}{dy} = 4x^3 + 6x^2y$ $N' = 4x^3 + 6x^2y$.

$\frac{\partial M'}{dy} = \frac{\partial N'}{dx}$ So it is exact.

Scanned by CamScanner

$$\int (4x^3y + 3x^2y^2 - x^3)\,dx + \int = C$$

$$\frac{4x^4y}{4} + \frac{3x^3y^2}{3} - \frac{x^4}{4} = C$$

$$x^4y + x^3y^2 - \frac{x^4}{4} = C.$$

* **Rule 5**: If $Mdx + Ndy = 0$ is of type $x^a y^b [my\,dx + nx\,dy] + x^{a'}y^{b'}[m'y\,dx + n'x\,dy] = 0$

then $I.F = x^h y^k$

where $\frac{a+h+1}{m} = \frac{b+k+1}{n}$ & $\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$

Ques: $(y^2 + 2x^2y)\,dx + [2x^3 - xy]\,dy = 0$

$M = y^2 + 2x^2y \qquad N = 2x^3 - xy$

$\frac{\partial M}{\partial y} = 2y + 2x^2 \qquad \frac{\partial N}{\partial x} = 6x^2 - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial N}$ — (1) is not exact.

$y^2dx + 2x^2y\,dx + 2x^3dy - xy\,dy = 0$

$x^0y\{y\,dx - x\,dy\} + x^2y^0\{2y\,dx + 2x\,dy\} = 0$

Compare it with

$x^ay^b\{my\,dx + nx\,dy\} + x^{a'}y^{b'}\{m'y\,dx + n'x\,dy\} = 0$

$a = 0,\ b = 1,\ m = 1,\ n = -1$

$a' = 2,\ b' = 0,\ m' = 2,\ n' = 2$

$I.F = x^h y^k$

where $\dfrac{a+h+1}{m} = \dfrac{b+k+1}{n}$

$\Rightarrow \dfrac{0+h+1}{1} = \dfrac{1+k+1}{-1}$

$\Rightarrow h+1 = -k-2$

$\Rightarrow h+k = -3$ —— (i)

agin $\dfrac{a'+h+1}{m'} = \dfrac{b'+k+1}{n'}$

$\Rightarrow \dfrac{2+h+1}{2} = \dfrac{0+k+1}{2} \Rightarrow h-k = -2$ —— (ii)

(ii) + (iii)

$2h = -5 \Rightarrow \boxed{h = -5/2}$ & $k = \cancel{-3h} -3-h$

$= -3 + 5/2 = -1/2$

$\boxed{k = \frac{-1}{2}}$

$I.F = x^h y^k = x^{-5/2} y^{-1/2} = \frac{1}{x^{5/2} y^{1/2}}$

multiply (i) with $\frac{1}{x^{5/2} y^{1/2}}$ $\{(y^2 + 2x^2 y)dx + (2x^3 - xy)dy = 0$

$\left[\frac{y^{3/2}}{x^{5/2}} + \frac{2y^{1/2}}{x^{1/2}}\right]dx + \left[\frac{2x^{1/2}}{y^{1/2}} - \frac{y^{1/2}}{x^{3/2}}\right]dy = 0$

Comp it $M'dx + N'dy = 0$

$M' = \frac{y^{3/2}}{x^{5/2}} + \frac{2y^{1/2}}{x^{1/2}}$ & $N' = \frac{2x^{1/2}}{y^{1/2}} - \frac{y^{1/2}}{x^{3/2}}$

$\frac{\partial M'}{\partial y} = \frac{3}{2}\frac{y^{1/2}}{x^{5/2}} + \frac{2 \times 1}{2}\frac{y^{-1/2}}{x^{1/2}}$

and

$\frac{\partial N'}{\partial x} = \frac{2 \times 1}{2} x - \left[\frac{-3}{2}\right] x^{-5/2}.$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow$ (ii) is exact

Scanned by CamScanner

$\therefore$ Solution to (ii) is $\int M' dx + \int$ [terms of N' without x] $dy = C$

$$= \int \left[\frac{y^{3/2}}{x^{3/2}} + \frac{2y^{1/2}}{x^{1/2}} \right] dx = C$$

$$y^{3/2} \frac{x^{-3/2}}{-3/2} + 2y^{1/2} \frac{x^{1/2}}{1/2} = C$$

$$= \frac{-2}{3} \frac{y^{3/2}}{x^{3/2}} + 4x^{1/2} y^{1/2} = C$$

Rule - VI (finding I.F by inspection.

$x\,dy + y\,dx = d(xy)$

$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

$\frac{x\,dy - y\,dx}{y^2} = -d\left(\frac{x}{y}\right)$

$\frac{x\,dy - y\,dx}{xy} = d\left[\log\left[\frac{y}{x}\right]\right]$

$\frac{x\,dy - y\,dx}{x^2+y^2} = d\left[\tan^{-1}\frac{y}{x}\right]$

$\frac{x\,dy - y\,dx}{x^2-y^2} = d\left[\frac{1}{2}\log\left[\frac{x+y}{x-y}\right]\right]$

Ques: $x\,dy - y\,dx = e^y (x^2+y^2)\,dy$.

$$\frac{x\,dy - y\,dx}{x^2+y^2} = e^y dy.$$

By Inspection rule.

$$d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = d\left[e^y\right]$$

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) = e^y + c}$$

Ques: $x\,dy - y\,dx + y^2 dx = 0$

Solⁿ: $\frac{x\,dy - y\,dx}{y^2} = -dx.$

$$\Rightarrow \frac{y\,dx - x\,dy}{y^2} = dx.$$

$$\int d\left(\frac{x}{y}\right) = \int dx.$$

↳ Integrating both side

$$\Rightarrow \frac{x}{y} = x + C.$$

Equation of first order and higher degree:

To solve diff. equation of first order and higher degree, we write $\frac{dy}{dx} = p$, then equation takes a form as.

$$F(x, y, p) = 0$$

Case-1: Equation Solvable for p:

Factorize (1), in linear factors as.

$$[p - f_1[x,y]][p - f_2[x,y]] \ldots = 0$$

$$p = f_1(x,y), \quad p = f_2[x,y] \ldots$$

$$p = f_1(x,y)$$

All these are diff. eq^n of first order and first degree.

Let solution to these equation is $f_1[x, y, c] = 0$

$$f_1(x,y,c) = 0, \quad f_2(x,y,c) = 0$$

Then general solutions of (1) will be.

$$f_1(x,y,c) \cdot f_2(x,y,c) = 0$$

Date ___/___/___
Page ___

Ques: $y\left(\frac{dy}{dx}\right)^2 - (x-y)\frac{dy}{dx} - x = 0$

Soln: This is equation of 1^{st} order & 2^{nd} degree.

Take $\frac{dy}{dx} = p$.

$\Rightarrow yp^2 - xp + yp - x = 0$

$\Rightarrow p(yp - x) + (yp - x) = 0$

$\Rightarrow (p+1)(yp - x) = 0$

$p + 1 = 0$

$\frac{dy}{dx} + 1 = 0$

$\Rightarrow dy = -dx$

$\Rightarrow$ Integrate

$y = -x + C_1$

$y + x - C_1 = 0$

$yp - x = 0$

$y\frac{dy}{dx} - x = 0$

$\rightarrow$ Integra $\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C_2$

$x^2 - y^2 + 2C_2 = 0.$

$\therefore$ General solution to starting eq is

$(y + x - C_1)\left[\frac{y^2}{2} - \frac{x^2}{2} - C_2\right] = 0$

Scanned by CamScanner

Date / /
Page

Ques: $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$, $p = \frac{dy}{dx}$

$p(p^2+2x) - y^2[p^2+x] = 0$

$(p-y^2)(p^2+x)$

$p[p^2 + 2xp - y^2p - 2xy^2] = 0$

$p\{p[p+2x] - y^2[p+x]\} = 0$

$p(p-y^2)(p+2x) = 0$

$\frac{dy}{dx} = 0$ | $\frac{dy}{dx} = y^2$ | $\frac{dy}{dx} = -2x$

$\frac{y^{-2+1}}{-2+1} = x + C_2$ | $y = \frac{-2x^2}{2} + C_3$

$y + C_1 = 0$ | $\frac{-1}{y} = x + C_2$ | $y = -x^2 + C_3$

$(y+C_1)\left(\frac{-1}{y} - x - C_2\right)\left(-x^2 + C_3 - y\right) = 0.$

H.W **Ques:** (i) $p(x$ $\cancel{xy}$ $\cdot p(p+y) = x(x+y)$

(ii) $y = x\{p + \sqrt{1+p^2}\}$

(iii) $xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0$

Scanned by CamScanner

Ans. (1) $p(p+y) = x(x+y)$

$\cancel{p^2 + py - x^2 - xy = 0}$

$\cancel{p(p+y) - x(x+y) = 0}$

$y[p-x] + [p^2 - x^2]$

$y[p-x] + (p+x)(p-x)$

$(p-x)[y+p+x] = 0$

$\frac{dy}{dx} = x$ | $y+p+x=0$

| $y + \frac{dy}{dx} + x = 0$

$y = \frac{x^2}{2} + C_1$ | $yx + y + \frac{x^2}{2} + C_2 = 0$

$\left(y - \frac{x^2}{2} - C_1\right)\left[yx + y + \frac{x^2}{2} + C_2\right) = 0$

★. Case : 2 [Equation solvable for y]

for a diff. eq. of first order & higher degree, we take $\frac{dy}{dx} = p$ then eq. takes a form. $f[x, y, p] = 0$ — (1)

If from above equation we can get

$$y = \phi[x, p]$$

diff. w.r.t x.

$$\frac{dy}{dx} = g\left[x, p, \frac{dp}{dx}\right]$$

$$p = g\left(x, p, \frac{dp}{dx}\right)$$

Solve this diff. in p & x, let $\psi(p, x) = c$ is the solution to above eq. — (ii)

∴ Eliminate p from (i) & (ii) to get general solution of eq. (1)

In case if we will not be able to eliminate p from eq. (1) & (ii) we simply (i) & (ii) together give us the general solution.

Date / /
Page

Ques: Solve $y - 2px = \tan^{-1}[p^2 x]$

Sol^n: $y = 2px + \tan^{-1}[p^2 x]$

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + \frac{1}{1+(p^2x)^2}\left[p^2 \cdot 1 + x \cdot 2p\frac{dp}{dx}\right]$$

$$p = 2p + 2x\frac{dp}{dx} + \frac{p^2}{1+p^4x^2} + \frac{2x\frac{dp}{dx}}{1+p^4x^2}$$

$$\Rightarrow -p - \frac{p^2}{1+p^4x^2} = \left[2x + \frac{2px}{1+p^4x^2}\right]\frac{dp}{dx}$$

$$\Rightarrow -p\left[1 + \frac{p}{1+p^4x^2}\right] = 2x\left[1 + \frac{p}{1+p^4x^2}\right]\frac{dp}{dx}$$

$$\Rightarrow 2x\left[1 + \frac{p}{1+p^4x^2}\right]\frac{dp}{dx} + p\left[1 + \frac{p}{1+p^4x^2}\right] = 0$$

$$\left[1 + \frac{p}{1+p^4x^2}\right]\underbrace{\left[2x\frac{dp}{dx} + p\right]}_{=0} = 0$$

$$2x\frac{dp}{dx} + p = 0$$

$\cancel{\int 2x\,dp = \int -p\,dx}$ $\qquad 2\frac{dx}{x} = -\frac{1}{p}dp$

$\cancel{2xp = -px + C}$ $\qquad \log p = \log x^{1/2} + \log C$

$$p = Cx^{1/2} \quad \text{—— (ii)}$$

Date / /
Page

Now, eliminate p from eq. (ii) to the original eq.

Put eq $p = cx^{-1/2}$ in (i).

$$y - 2cx^{-1/2}x = \tan^{-1}[xc^2x^{-1}]$$

$y - 2c\sqrt{x} = \tan^{-1}[c^2]$ is the general soln. given in (i).

Ques: Solve $y = x + a\tan^{-1}p$ — (i).

$$\frac{dy}{dx} = 1 + a\frac{1}{1+p^2}\frac{dp}{dx}$$

$$p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$$

$$(p-1) = \frac{a}{1+p^2}\frac{dp}{dx}$$

$$\frac{a\,dp}{(1+p^2)(p-1)} = dx$$

Integrating.

$$a\int\frac{dp}{(p^2+1)(p-1)} = \int dx + C$$

→ partial fraction.

Now, $$\frac{1}{(p^2+1)(p-1)} = \frac{A}{p-1} + \frac{Bp+D}{p^2+1}.$$

Date / /
Page

$$\Rightarrow \quad 1 = A(p^2+1) + (Bp + D)(p-1).$$

$\boxed{A = \frac{+1}{2}}$ $\boxed{D = \frac{-1}{2}}$ $\boxed{B = \frac{-1}{2}}$

$$= \frac{1}{2}\frac{1}{p+1} - \frac{\frac{1}{2}p - 1/2}{p^2+1}.$$

$$= \frac{1}{2}\log(p-1) - \frac{1}{4}\log|p^2+1| - \frac{1}{2}\tan^{-1}p.$$

$$a\left\{\frac{1}{2}\log(p-1) - \frac{1}{4}\log|p^2+1| - \frac{1}{2}\tan^{-1}p\right. = x + C. \quad \text{(III)}$$

Now, replace p from eq. (III) & (I).

$$y = x + a\tan^{-1}p \Rightarrow \frac{y-x}{a} = \tan^{-1}p$$

$$p = \tan\left(\frac{y-x}{a}\right)$$

$$a\left\{\frac{1}{2}\log\left[\tan\left(\frac{y-x}{a}\right) - 1\right] - \frac{1}{4}\log\left[\tan^{-1}\left(\frac{y+x}{a}\right) + 1\right]\right.$$

$$- \frac{1}{2}\left(\frac{y-x}{a}\right) = x + C.$$

Scanned by CamScanner

Date / /
Page

* Rule - 3 : Equation solvable for x :

let equation with first order & higher degree because $f[x, y, p] = 0$

If from above equation, we can get

$x = \phi[y, p]$ diff. w.r.t y.

$$\Rightarrow \frac{dx}{dy} = g\left[y, p, \frac{dp}{dy}\right]$$

$$\Rightarrow \frac{1}{p} = g\left[y, p, \frac{dp}{dy}\right]$$

solve this differential eq. &

let the solution is

$\psi(p, y, c) = 0$ — (II)

then eliminate p from (I) & (II) to get the general solution.

H.W

Ans: Solve, $y = px + x^4p^2$

$$x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$$

Ques: Solve $y = 2px + y^2p^3$

Sol^n $\rightarrow$ $x = \frac{y - y^2p^3}{2p}$

diff w.r.t y.

$$\frac{dx}{dy} = \frac{2p\{1 - y^2 3p^2\frac{dp}{dy} - p^3 2y - (y - y^2p^3) \cdot 2\frac{dp}{dy}}{4p^2}$$

$$\Rightarrow \frac{1}{p} = \frac{2p - 6p^3y^2\frac{dp}{dy} - 4p^4y - 2y\frac{dp}{dy} + 2y^2p^3\frac{dp}{dy}}{4p^2}$$

$$4p = 2p - 6p^3y^2\frac{dp}{dy} - 4p^4y - 2y\frac{dp}{dy} + 2y^2p\frac{dp}{dy}.$$

$$\Rightarrow 2p + 4p^4y = -4p^3y^2\frac{dp}{dy} - 2y\frac{dp}{dy}$$

$$\Rightarrow 2p[1 + 2p^3y] = -2y[1 + 2p^3y]\frac{dp}{dy}$$

$$\Rightarrow 2p\{1 + 2p^3y\} + 2y[1 + 2p^3y]\frac{dp}{dy} = 0$$

$$\Rightarrow 2(1 + 2p^3y)\left(p + y\frac{dp}{dy}\right) = 0$$

Date / /
Page

$p + \frac{y\,dp}{dy} = 0 \Rightarrow p = -\frac{y\,dp}{dy} \Rightarrow \frac{dy}{y} = -\frac{dp}{p}$

Integrate.

$\log p = -\log y + \log c$

$\Rightarrow \log p = \log\left(\frac{c}{y}\right) \Rightarrow \boxed{p = \frac{c}{y}}$ — (ii)

Eliminating p from (i) & (ii), we get

$y = 2\left(\frac{c}{y}\right)x + \left[\frac{c}{y}\right]^3 y^2 = \frac{2cx}{y} + \frac{c^3}{y}$

$y^2 = 2cx + c^3$ is the general solution for - (i).

Clairaut's differential Equation:

An equation of type $y = px + f[p]$ is called clairaut's equation and general solution to the eq. is

$y = cx + f(c)$, where c is an arbitrary constant.

Ques: Solve $xp^2 - yp + a = 0$

Sol^n $xp^2 + a = yp \Rightarrow y = px + \frac{a}{p}$

This is clairaut form. So,

Solution is $y = cx + \frac{a}{c}$

Ques: $p = \log(px - y)$

Sol^n $px - y = e^p \Rightarrow y = \underbrace{px - e^p}$

↳ It is clairaut eq.

∴ general sol^n is $y = cx - e^c$.

H.W

Ques: solve $y = px + \sqrt{a^2p^2 + b^2}$

Ques: solve $\sin px \cos y = \cos px \sin y + p$

Sol^n $\sin px \cos y - \cos px \sin y = p$

$\sin(px - y) = p \Rightarrow px - y = \sin^{-1} p$

$y = px - \sin^{-1} p$, It's clairaut form.

∴ General solution is $y = cx - \sin^{-1} c$.